HYPERSURFACES OF CONSTANT CURVATURE IN HYPERBOLIC SPACE.

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ABSTRACT. We show that for a very general and natural class of curvature functions, the problem of finding a complete strictly convex hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma \in (0,1)$ with a prescribed asymptotic boundary Γ at infinity has at least one solution which is a "vertical graph" over the interior (or the exterior) of Γ. There is uniqueness for a certain subclass of these curvature functions which includes the curvature quotients $(\frac{H_n}{H_l})^{\frac{1}{n-l}}$, $l-2$, or $l-1$. For smooth simple Γ, as σ varies between 0 and 1, these hypersurfaces foliate the two components of the complement of the hyperbolic convex hull of Γ.

1. Introduction

In this paper we return to our earlier study [7] of complete locally strictly convex hypersurfaces of constant curvature in hyperbolic space $\mathbb{H}^{n+1}$ with a prescribed asymptotic boundary at infinity. Given $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$ and a smooth symmetric function f of n variables, we seek a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{1.1}$$

$$\partial\Sigma = \Gamma \tag{1.2}$$

where $\kappa[\Sigma] = (\kappa_1, \ldots, \kappa_n)$ denotes the *positive* hyperbolic principal curvatures of Σ and $\sigma \in (0,1)$ is a constant.

We will use the half-space model,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{\sum_{i=1}^{n+1} dx_i^2}{x_{n+1}^2}. \tag{1.3}$$

Thus $\partial_\infty \mathbb{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (1.2) may be understood in the Euclidean sense. For convenience we say Σ has compact asymptotic boundary if $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ is compact with respect to the Euclidean metric in $\mathbb{R}^n$.

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The function f is assumed to satisfy the fundamental structure conditions in

$$K_n^+ := \left\{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\right\} : \tag{1.4}$$

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \ \text{ in } K_n^+, \ \ 1 \leq i \leq n, \tag{1.5}$$

$$f \text{ is a concave function in } K_n^+, \tag{1.6}$$

and

$$f > 0 \ \text{ in } K_n^+, \ \ f = 0 \ \text{ on } \partial K_n^+ \tag{1.7}$$

In addition, we shall assume that f is normalized

$$f(1, \ldots, 1) = 1 \tag{1.8}$$

and satisfies the following more technical assumptions

$$f \text{ is homogeneous of degree one} \tag{1.9}$$

and

$$\lim_{R \to +\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \geq 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{1.10}$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

All these assumptions are satisfied by $f = (H_n/H_l)^{\frac{1}{n-l}}$, $0 \leq l < n$, where H_l is the normalized l-th elementary symmetric polynomial ($H_0 = 1$, $H_1 = H$ and $H_n = K$ the mean and extrinsic Gauss curvatures, respectively). See [2] for proof of (1.5) and (1.6). For (1.10) one easily computes that

$$\lim_{R \to +\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) = \binom{n}{l}^{\frac{1}{n-l}}.$$

Moreover, if g^k, $k = 1, \ldots N$ satisfy (1.5)-(1.10), then so does the "concave sum" $f = \sum_{k=1}^N \alpha_k g^k$ or "concave product" $f = \Pi_{k=1}^N (g^k)^{\alpha_k}$ where $\alpha_k > 0$, $\sum_{k=1}^N \alpha_k = 1$.

Since f is symmetric, by (1.6), (1.8) and (1.9) we have

$$f(\lambda) \leq f(\mathbf{1}) + \sum f_i(\mathbf{1})(\lambda_i - 1) = \sum f_i(\mathbf{1})\lambda_i = \frac{1}{n} \sum \lambda_i \ \text{ in } K_n^+ \tag{1.11}$$

and

$$\sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)(1 - \lambda_i) \geq f(\mathbf{1}) = 1 \ \text{ in } K_n^+. \tag{1.12}$$

In this paper all hypersurfaces in $\mathbb{H}^{n+1}$ we consider are assumed to be connected and orientable. If Σ is a complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field of Σ is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+ \setminus \Sigma$, and the (both hyperbolic and Euclidean) principal curvatures of Σ are calculated with respect to this normal vector field.

As in our earlier work [11, 10, 5, 7, 6], we will take $\Gamma = \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a smooth domain and seek Σ as the graph of a function $u(x)$ over Ω, i.e.

$$\Sigma = \{(x, x_{n+1}) : x \in \Omega,\ x_{n+1} = u(x)\}.$$

Then the coordinate vector fields and upper unit normal are given by

$$X_i = e_i + u_i e_{n+1},\ \ \mathbf{n} = u\nu = u\frac{(-u_i e_i + e_{n+1})}{w},$$

where $w = \sqrt{1 + |\nabla u|^2}$. The first fundamental form g_{ij} is then given by

$$g_{ij} = \langle X_i, X_j\rangle = \frac{1}{u^2}(\delta_{ij} + u_i u_j) = \frac{g^e_{ij}}{u^2}\ . \tag{1.13}$$

To compute the second fundamental form h_{ij} we use

$$\Gamma^k_{ij} = \frac{1}{x_{n+1}}\{-\delta_{jk}\delta_{in+1} - \delta_{ik}\delta_{jn+1} + \delta_{ij}\delta_{kn+1}\} \tag{1.14}$$

to obtain

$$\nabla_{X_i} X_j = (\frac{\delta_{ij}}{x_{n+1}} + u_{ij} - \frac{u_i u_j}{x_{n+1}})e_{n+1} - \frac{u_j e_i + u_i e_j}{x_{n+1}}\ . \tag{1.15}$$

Then

$$\begin{aligned} h_{ij} &= \langle \nabla_{X_i} X_j, u\nu\rangle = \frac{1}{uw}(\frac{\delta_{ij}}{u} + u_{ij} - \frac{u_i u_j}{u} + 2\frac{u_i u_j}{u}) \\ &= \frac{1}{u^2 w}(\delta_{ij} + u_i u_j + u u_{ij}) = \frac{h^e_{ij}}{u} + \frac{\nu^{n+1}}{u^2} g^e_{ij}. \end{aligned} \tag{1.16}$$

The hyperbolic principal curvatures κ_i of Σ are the roots of the characteristic equation

$$\det(h_{ij} - \kappa g_{ij}) = u^{-n}\det(h^e_{ij} - \frac{1}{u}(\kappa - \frac{1}{w})g^e_{ij}) = 0.$$

Therefore,

$$\kappa_i = u\kappa^e_i + \nu^{n+1}. \tag{1.17}$$

The relations (1.16) and (1.17) are easily seen to hold for parametric hypersurfaces.

One beautiful consequence of (1.16) is the following result of [7].

Theorem 1.1. *Let Σ be a complete locally strictly convex C^2 hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity. Then Σ is the (vertical) graph of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $u > 0$ in Ω and $u = 0$ on $\overline{\Omega}$, for some domain $\Omega \subset \mathbb{R}^n$:*

$$\Sigma = \big\{(x, u(x)) \in \mathbb{R}^{n+1}_+ : x \in \Omega\big\}$$

such that

$$\{\delta_{ij} + u_i u_j + u u_{ij}\} > 0 \ \text{ in } \Omega. \tag{1.18}$$

That is, the function $u^2 + |x|^2$ is strictly convex.

According to Theorem 1.1, our assumption that Σ is a graph is completely general and the asymptotic boundary Γ must be the boundary of some bounded domain Ω in $\mathbb{R}^n$.

Problem (1.1)-(1.2) then reduces to the Dirichlet problem for a fully nonlinear second order equation which we shall write in the form

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1.19}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.20}$$

We seek solutions of equation (1.19) satisfying (1.18). Following the literature we call such solutions *admissible*. By [2] condition (1.5) implies that equation (1.19) is elliptic for admissible solutions. Our goal is to show that the Dirichlet problem (1.19)-(1.20) admits smooth admissible solutions for all $0 < \sigma < 1$, which is optimal.

Our main result of the paper may be stated as follows.

Theorem 1.2. *Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose $\sigma \in (0, 1)$ and that f satisfies conditions (1.5)-(1.10) in K_n^+. Then there exists a complete locally strictly convex hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1.1)-(1.2) with uniformly bounded principal curvatures*

$$|\kappa[\Sigma]| \le C \ \text{ on } \Sigma. \tag{1.21}$$

Moreover, Σ is the graph of an admissible solution $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ of the Dirichlet problem (1.19)-(1.20). *Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and*

$$\begin{aligned} u|D^2u| &\le C \quad \text{in } \Omega, \\ \sqrt{1+|Du|^2} &= \frac{1}{\sigma} \quad \text{on } \partial\Omega \end{aligned} \tag{1.22}$$

For Gauss curvature, $f(\lambda) = (H_n)^{\frac{1}{n}}$, Theorem 1.2 was proved by Rosenberg and Spruck [11].

Equation (1.19) is singular where $u = 0$. It is therefore natural to approximate the boundary condition (1.20) by

$$u = \epsilon > 0 \quad \text{on } \partial\Omega. \tag{1.23}$$

When ϵ is sufficiently small, we showed in [7] that the Dirichlet problem (1.19),(1.23) is solvable for all $\sigma \in (0, 1)$.

Theorem 1.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $\sigma \in (0, 1)$. Suppose f satisfies (1.5)-(1.10) in K_n^+. Then for any $\epsilon > 0$ sufficiently small, there exists an admissible solution $u^\epsilon \in C^\infty(\bar{\Omega})$ of the Dirichlet problem (1.19), (1.23). Moreover, u^ϵ satisfies the* a priori *estimates*

$$\sqrt{1+|Du^\epsilon|^2} \le \frac{1}{\sigma} + C\epsilon,\ u^\epsilon|D^2u^\epsilon| \le C \quad \text{on } \partial\Omega\,, \tag{1.24}$$

and

$$u^\epsilon|D^2u^\epsilon| \le \frac{C}{\epsilon^2} \quad \text{in } \Omega. \tag{1.25}$$

where C is independent of ϵ.

Remark 1.4. The global gradient estimate, Lemma 3.4 of [7] is not correct as stated. This may be corrected using the convexity argument of [11] or using Corollary 3.3 of section 3 of this paper. Theorem 1.3 above as well as Theorem 1.2 of [7] remain valid. However no apriori uniqueness can be asserted. In Theorem 1.6 we prove a uniqueness result for a special class of curvature functions.

Our main technical difficulty in proving Theorem 1.2 is that the estimate (1.25) does not allow us to pass to the limit. In [7] we were able to obtain a global estimate independent of ϵ for the hyperbolic principal curvatures for $\sigma^2 > \frac{1}{8}$. In this paper we obtain such estimates for all $\sigma \in (0, 1)$ by proving a maximum principle for the largest hyperbolic principal curvature.

Theorem 1.5. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $\sigma \in (0,1)$. Suppose f satisfies (1.5)-(1.10) in K_n^+. Then for any admissible solution $u^\epsilon \in C^\infty(\bar{\Omega})$ of the Dirichlet problem (1.19), (1.23),*

$$\max_{\mathbf{x}\in\Sigma^\epsilon} \kappa_{\max}(\mathbf{x}) \leq C(1 + \max_{\mathbf{x}\in\partial\Sigma^\epsilon} \kappa_{\max}(\mathbf{x})) \tag{1.26}$$

where $\Sigma^\epsilon =$ graph u^ϵ and C is independent of ϵ.

By Theorem 1.5, the hyperbolic principal curvatures of the admissible solution u^ϵ given in Theorem 1.3 are uniformly bounded above independent of ϵ. Since $f(\kappa[u^\epsilon]) = \sigma$ and $f = 0$ on ∂K_n^+, the hyperbolic principal curvatures admit a uniform positive lower bound independent of ϵ and therefore (1.19) is uniformly elliptic on compact subsets of Ω for the solution u^ϵ. By the interior estimates of Evans and Krylov, we obtain uniform $C^{2,\alpha}$ estimates for any compact subdomain of Ω. The proof of Theorem 1.2 is now routine.

Finally we prove a uniqueness result and as an application prove a result about foliations. This latter result is relevant to the study of foliations of the complement of the convex core of quasi-fuchsian manifolds (see [8], [11], [13]).

Theorem 1.6. *Suppose f satisfies (1.5)-(1.10) in K_n^+ and in addition,*

$$\sum f_i > \sum \lambda_i^2 f_i \text{ in } K_n^+ \cap \{0 < f < 1\}. \tag{1.27}$$

Then the solutions given in Theorem 1.2 and Theorem 1.3 are unique. In particular uniqueness holds for $f = (\frac{H_n}{H_l})^{\frac{1}{n-l}}$ with $l - 1$ or $l - 2$.

Theorem 1.7. *a. Let f satisfy the conditions of Theorem 1.6 and assume that Γ is smooth. Then for each $\sigma \in (0,1)$ there are exactly two embedded strictly locally convex hypersurfaces satisfying (1.1), (1.2). Each surface is a graph of $u^\sigma \in C^\infty(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ where $\Omega^\pm$ are the components of the complement of Γ. Moreover the solution hypersurfaces $\Sigma^\sigma =$ graph u^σ have uniformly bounded principal curvatures and foliate each component of $\mathbb{H}^{n+1} \setminus \mathcal{CH}(\Gamma)$, the complement of the hyperbolic convex hull of Γ.*

b. Let $f = \frac{H_n}{H_{n-1}}$ and $\Gamma = \partial\Omega$ where Ω is a simply connected Jordan domain. If $n > 2$, assume in addition that Γ is regular for Laplace's equation. Then the conclusions of part a. hold with $u^\sigma(x) \in C^\infty(\Omega^\pm) \cap C^0(\overline{\Omega^\pm})$ and the principal curvatures are uniformly bounded on compact subsets.

Remark 1.8. i. Graham Smith pointed out to us that in the special case $n = 3$, $f = (\frac{K}{H})^{\frac{1}{2}}$ is his special Lagrangian curvature with angle $\theta = \pi$ and interior curvature bounds follow from the geometric ideas of his paper [14]. Moreover in Lemma 7.4 of [13] he showed that special Lagrangian curvature with angle $\theta \geq (n-1)\frac{\pi}{2}$ satisfies our uniqueness condition (1.27). Thus by Theorems 1.6 and 1.7, the existence of foliations of constant special Lagrangian curvature can be proven for $\theta \geq (n-1)\frac{\pi}{2}$ for all n. This includes the special case $f = K^{\frac{1}{2}}$ when $n = 2$ and $f = (\frac{K}{H})^{\frac{1}{3}}$ for $n = 3$ mentioned above. See [15] for Graham Smith's most recent work which has some overlap with ours.

ii. Rosenberg and Spruck [11] proved part b of Theorem 1.7 for $f = K^{\frac{1}{2}}$ in case $n = 2$. Here and also in [11], no claim is made about the higher regularity of the $\mathcal{CH}(\Gamma)$. In other words, the curvature estimates obtained in the proof of Theorem 1.7 (global for Γ smooth and interior for Γ Jordan) blow up as $\sigma \to 0$. We have not yet derived interior curvature estimates for the case $f = \frac{H_n}{H_{n-2}}$ in the general case.

The organization of the paper is as follows. In section 2 we establish some basic identities on a hypersurface Σ satisfying (1.1) that will form the basis of the global gradient estimates derived in section 3 and the maximum principle for $\kappa_{\max}$, the largest principal curvature of Σ, which is carried out in section 4. Finally in section 5 we prove the uniqueness Theorem 1.6 and the foliation Theorem 1.7.

2. Formulas on hypersurfaces

In this section we will derive some basic identities on a hypersurface by comparing the induced hyperbolic and Euclidean metrics.

Let Σ be a hypersurface in $\mathbb{H}^{n+1}$. We shall use g and ∇ to denote the induced hyperbolic metric and Levi-Civita connections on Σ, respectively. As Σ is also a submanifold of $\mathbb{R}^{n+1}$, we shall usually distinguish a geometric quantity with respect to the Euclidean metric by adding a 'tilde' over the corresponding hyperbolic quantity. For instance, $\tilde{g}$ denotes the induced metric on Σ from $\mathbb{R}^{n+1}$, and $\tilde{\nabla}$ is its Levi-Civita connection.

Let $\mathbf{x}$ be the position vector of Σ in $\mathbb{R}^{n+1}$ and set

$$u = \mathbf{x} \cdot \mathbf{e}$$

where $\mathbf{e}$ is the unit vector in the positive x_{n+1} direction in $\mathbb{R}^{n+1}$, and '$\cdot$' denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. We refer u as the *height function* of Σ.

Throughout the paper we assume Σ is orientable and let $\mathbf{n}$ be a (global) unit normal vector field to Σ with respect to the hyperbolic metric. This also determines a unit normal ν to Σ with respect to the Euclidean metric by the relation

$$\nu = \frac{\mathbf{n}}{u}.$$

We denote $\nu^{n+1} = \mathbf{e} \cdot \nu$.

Let $(z_1, \ldots, z_n)$ be local coordinates and

$$\tau_i = \frac{\partial}{\partial z_i}, \quad i = 1, \ldots, n.$$

The hyperbolic and Euclidean metrics of Σ are given by

$$g_{ij} = \langle \tau_i, \tau_j \rangle, \quad \tilde{g}_{ij} = \tau_i \cdot \tau_j = u^2 g_{ij},$$

while the second fundamental forms are

$$\begin{aligned} h_{ij} &= \langle D_{\tau_i} \tau_j, \mathbf{n} \rangle = -\langle D_{\tau_i} \mathbf{n}, \tau_j \rangle, \\ \tilde{h}_{ij} &= \nu \cdot \tilde{D}_{\tau_i} \tau_j = -\tau_j \cdot \tilde{D}_{\tau_i} \nu, \end{aligned} \tag{2.1}$$

where D and $\tilde{D}$ denote the Levi-Civita connection of $\mathbb{H}^{n+1}$ and $\mathbb{R}^{n+1}$, respectively. The following relations are well known (see (1.16), (1.17)):

$$h_{ij} = \frac{1}{u} \tilde{h}_{ij} + \frac{\nu^{n+1}}{u^2} \tilde{g}_{ij}. \tag{2.2}$$

and $\tilde{\kappa}_1, \cdots, \tilde{\kappa}_n$ by the formula

$$\kappa_i = u \tilde{\kappa}_i + \nu^{n+1}, \quad i = 1, \cdots, n. \tag{2.3}$$

The Christoffel symbols are related by the formula

$$\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k - \frac{1}{u} (u_i \delta_{kj} + u_j \delta_{ik} - \tilde{g}^{kl} u_l \tilde{g}_{ij}). \tag{2.4}$$

It follows that for $v \in C^2(\Sigma)$

$$\nabla_{ij} v = v_{ij} - \Gamma_{ij}^k v_k = \tilde{\nabla}_{ij} v + \frac{1}{u} (u_i v_j + u_j v_i - \tilde{g}^{kl} u_k v_l \tilde{g}_{ij}) \tag{2.5}$$

where (and in sequel)

$$v_i = \frac{\partial v}{\partial x_i}, \; v_{ij} = \frac{\partial^2 v}{\partial x_i x_j}, \text{ etc.}$$

In particular,

$$\nabla_{ij} u = \tilde{\nabla}_{ij} u + \frac{2 u_i u_j}{u} - \frac{1}{u} \tilde{g}^{kl} u_k u_l \tilde{g}_{ij} \tag{2.6}$$

and

$$\nabla_{ij}\frac{1}{u} = -\frac{1}{u^2}\tilde{\nabla}_{ij}u + \frac{1}{u^3}\tilde{g}^{kl}u_k u_l \tilde{g}_{ij}. \tag{2.7}$$

Moreover,

$$\nabla_{ij}\frac{v}{u} = v\nabla_{ij}\frac{1}{u} + \frac{1}{u}\tilde{\nabla}_{ij}v - \frac{1}{u^2}\tilde{g}^{kl}u_k v_l \tilde{g}_{ij}. \tag{2.8}$$

In $\mathbb{R}^{n+1}$,

$$\begin{aligned} \tilde{g}^{kl}u_k u_l &= |\tilde{\nabla}u|^2 = 1 - (\nu^{n+1})^2 \\ \tilde{\nabla}_{ij}u &= \tilde{h}_{ij}\nu^{n+1}. \end{aligned} \tag{2.9}$$

Therefore, by (2.3) and (2.7),

$$\begin{aligned} \nabla_{ij}\frac{1}{u} &= -\frac{\nu^{n+1}}{u^2}\tilde{h}_{ij} + \frac{1}{u^3}(1 - (\nu^{n+1})^2)\tilde{g}_{ij} \\ &= \frac{1}{u}(g_{ij} - \nu^{n+1}h_{ij}). \end{aligned} \tag{2.10}$$

We note that (2.8) and (2.10) still hold for general local frames $\tau_1, \ldots, \tau_n$. In particular, if $\tau_1, \ldots, \tau_n$ are orthonormal in the hyperbolic metric, then $g_{ij} = \delta_{ij}$ and $\tilde{g}_{ij} = u^2\delta_{ij}$.

We now consider equation (1.1) on Σ. Let $\mathcal{A}$ be the vector space of $n \times n$ matrices and

$$\mathcal{A}^+ = \{A = \{a_{ij}\} \in \mathcal{A} : \lambda(A) \in K_n^+\},$$

where $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of A. Let F be the function defined by

$$F(A) = f(\lambda(A)), \quad A \in \mathcal{A}^+ \tag{2.11}$$

and denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij}\partial a_{kl}}(A). \tag{2.12}$$

Since $F(A)$ depends only on the eigenvalues of A, if A is symmetric then so is the matrix $\{F^{ij}(A)\}$. Moreover,

$$F^{ij}(A) = f_i\delta_{ij}$$

when A is diagonal, and

$$F^{ij}(A)a_{ij} = \sum f_i(\lambda(A))\lambda_i = F(A), \tag{2.13}$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \tag{2.14}$$

Equation (1.1) can therefore be rewritten in a local frame $\tau_1, \ldots, \tau_n$ in the form

$$F(A[\Sigma]) = \sigma \tag{2.15}$$

where $A[\Sigma] = \{g^{ik}h_{kj}\}$. Let $F^{ij} = F^{ij}(A[\Sigma])$, $F^{ij,kl} = F^{ij,kl}(A[\Sigma])$.

Lemma 2.1. *Let Σ be a smooth hypersurface in $\mathbb{H}^{n+1}$ satisfying equation* (1.1). *Then in a local orthonormal frame,*

$$F^{ij}\nabla_{ij}\frac{1}{u} = -\frac{\sigma\nu^{n+1}}{u} + \frac{1}{u}\sum f_i. \tag{2.16}$$

and

$$F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} = \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2. \tag{2.17}$$

Proof. The first identity follows immediately from (2.10), (2.13) and assumption (1.9). To prove (2.17) we recall the identities in $\mathbb{R}^{n+1}$

$$\begin{aligned} (\nu^{n+1})_i &= -\tilde{h}_{ij}\tilde{g}^{jk}u_k, \\ \tilde{\nabla}_{ij}\nu^{n+1} &= -\tilde{g}^{kl}(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij}). \end{aligned} \tag{2.18}$$

By (2.2), (2.13), (2.14), and $\tilde{g}^{ik} = \delta_{jk}/u^2$ we see that

$$\begin{aligned} F^{ij}\tilde{g}^{kl}\tilde{h}_{il}\tilde{h}_{kj} &= \frac{1}{u^2}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} \\ &= F^{ij}(h_{ik}h_{kj} - 2\nu^{n+1}h_{ij} + (\nu^{n+1})^2\delta_{ij}) \\ &= f_i\kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2\sum f_i. \end{aligned} \tag{2.19}$$

As a hypersurface in $\mathbb{R}^{n+1}$, it follows from (2.3) that Σ satisfies

$$f(u\tilde{\kappa}_1 + \nu^{n+1}, \ldots, u\tilde{\kappa}_n + \nu^{n+1}) = \sigma,$$

or equivalently,

$$F(\{\tilde{g}^{ik}(u\tilde{h}_{kj} + \nu^{n+1}\tilde{g}_{kj})\}) = \sigma. \tag{2.20}$$

Differentiating equation (2.20) and using $\tilde{g}_{ik} = u^2\delta_{ik}$, $\tilde{g}^{ik} = \delta_{ik}/u^2$, we obtain

$$F^{ij}(u\tilde{\nabla}_k\tilde{h}_{ij} + u_k\tilde{h}_{ij} + (\nu^{n+1})_k u^2\delta_{ij}) = 0. \tag{2.21}$$

That is,

$$\begin{aligned} F^{ij}\tilde{\nabla}_k\tilde{h}_{ij} + (\nu^{n+1})_k u\sum F^{ii} &= -\frac{u_k}{u}F^{ij}\tilde{h}_{ij} \\ &= -u_kF^{ij}(h_{ij} - \nu^{n+1}\delta_{ij}) \\ &= -u_k\Big(\sigma - \nu^{n+1}\sum f_i\Big). \end{aligned} \tag{2.22}$$

Finally, combining (2.8), (2.16), (2.18), (2.19), (2.22), and the first identity in (2.9), we derive

$$
\begin{aligned}
F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} &= \nu^{n+1}F^{ij}\nabla_{ij}\frac{1}{u} + \frac{|\tilde{\nabla} u|^2}{u}F^{ij}\tilde{h}_{ij} - \frac{\nu^{n+1}}{u^3}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} \\
&= \frac{\nu^{n+1}}{u}\Big(\sum f_i - \nu^{n+1}\sigma\Big) + \frac{|\tilde{\nabla} u|^2}{u}\Big(\sigma - \nu^{n+1}\sum f_i\Big) \\
&\quad - \frac{\nu^{n+1}}{u}\Big(f_i\kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2\sum f_i\Big) \\
&= \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2.
\end{aligned}
\tag{2.23}
$$

This proves (2.17).

□

3. The asymptotic angle maximum principle and gradient estimates

In this section we show that the upward unit normal of a solution tends to a fixed asymptotic angle on approach to the boundary. This implies a global gradient bound on solutions.

Theorem 3.1. *Let Σ be a smooth strictly locally convex hypersurface in $\mathbb{H}^{n+1}$ satisfying equation* (1.1)*. Suppose Σ is globally a graph:*

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

where Ω is a domain in $\mathbb{R}^n \equiv \partial\mathbb{H}^{n+1}$. Then

$$F^{ij}\nabla_{ij}\frac{\sigma - \nu^{n+1}}{u} \geq \sigma(1-\sigma)\frac{(\sum f_i - 1)}{u} \geq 0 \tag{3.1}$$

and so,

$$\frac{\sigma - \nu^{n+1}}{u} \leq \sup_{\partial\Sigma}\frac{\sigma - \nu^{n+1}}{u} \quad \text{on } \Sigma. \tag{3.2}$$

Moreover, if $u = \epsilon > 0$ on $\partial\Omega$, then there exists $\epsilon_0 > 0$ depending only on $\partial\Omega$, such that for all $\epsilon \leq \epsilon_0$,

$$\frac{\sigma - \nu^{n+1}}{u} \leq \frac{\sqrt{1-\sigma^2}}{r_1} + \frac{\varepsilon(1+\sigma)}{r_1^2} \quad \text{on } \Sigma \tag{3.3}$$

where r_1 is the maximal radius of exterior tangent spheres to $\partial\Omega$.

Proof. Set $\eta = \frac{\sigma - \nu^{n+1}}{u}$. By (2.16) and (2.17) we have

$$F^{ij}\nabla_{ij}\eta = \frac{\sigma}{u}\Big(\sum f_i - 1\Big) + \frac{\nu^{n+1}}{u}\Big(\sum f_i\kappa_i^2 - \sigma^2\Big).$$

On the other hand,

$$\sum \kappa_i^2 f_i \geq \frac{(\sum \kappa_i f_i)^2}{\sum f_i} = \frac{\sigma^2}{\sum f_i}.$$

Hence,

$$F^{ij}\nabla_{ij}\eta \geq \frac{\sigma}{u}\Big(\sum f_i - 1\Big)\Big(1 - \frac{\sigma\nu^{n+1}}{\sum f_i}\Big) \geq \frac{\sigma(1-\sigma)}{u}\Big(\sum f_i - 1\Big) \geq 0.$$

So (3.2) follows from the maximum principle, while (3.3) follows from (3.2) and the approximate asymptotic angle condition,

$$\eta \leq \frac{\sqrt{1-\sigma^2}}{r_1} + \frac{\varepsilon(1+\sigma)}{r_1^2} \text{ on } \partial\Sigma$$

which is proved in Lemma 3.2 of [7]. ∎

Proposition 3.2. *Let Σ be a smooth strictly locally convex graph*

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

in $\mathbb{H}^{n+1}$ satisfying $u \geq \varepsilon$ in Ω, $u = \varepsilon$ on $\partial\Omega$. Then

$$\frac{1}{\nu^{n+1}} \leq \max\Big\{\frac{\max_\Omega u}{u}, \max_{\partial\Omega}\frac{1}{\nu^{n+1}}\Big\}. \tag{3.4}$$

Proof. Let $h = \frac{u}{\nu^{n+1}} = uw$ and suppose that h assumes its maximum at an interior point x_0. Then at x_0,

$$\partial_i h = u_i w + u\frac{u_k u_{ki}}{w} = (\delta_{ki} + u_k u_i + u u_{ki})\frac{u_k}{w} = 0 \ \ \forall\, 1 \leq i \leq n.$$

Since Σ is strictly locally convex, this implies that $\nabla u = 0$ at x_0 so the proposition follows immediately. ∎

Combining Theorem 3.1 and Proposition 3.2 gives

Corollary 3.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $\sigma \in (0,1)$. Suppose f satisfies (1.5)-(1.10) in K_n^+. Then for any $\epsilon > 0$ sufficiently small, any admissible solution $u^\epsilon \in C^\infty(\bar{\Omega})$ of the Dirichlet problem (1.19),(1.23) satisfies the* apriori *estimate*

$$|\nabla u^\epsilon| \leq C \quad \text{in } \Omega \tag{3.5}$$

where C is independent of ϵ.

4. Curvature estimates

In this section we prove a maximum principal for the largest principal curvature of locally strictly convex graphs satisfying $f(\kappa) = \sigma$.

Let Σ be a smooth hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$. For a fixed point $\mathbf{x}_0 \in \Sigma$ we choose a local orthonormal frame $\tau_1, \ldots, \tau_n$ around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$. The calculations below are done at $\mathbf{x}_0$. For convenience we shall write $v_{ij} = \nabla_{ij} v$, $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_{lk} h_{ij} = \nabla_l \nabla_k h_{ij}$, etc.

Since $\mathbb{H}^{n+1}$ has constant sectional curvature -1, by the Codazzi and Gauss equations we have $h_{ijk} = h_{ikj}$ and

$$
\begin{aligned}
h_{iijj} &= h_{jjii} + (h_{ii} h_{jj} - 1)(h_{ii} - h_{jj}) \\
&= h_{jjii} + (\kappa_i \kappa_j - 1)(\kappa_i - \kappa_j).
\end{aligned} \tag{4.1}
$$

Consequently for each fixed j,

$$
F^{ii} h_{jjii} = F^{ii} h_{iijj} + (1 + \kappa_j^2) \sum f_i \kappa_i - \kappa_j \sum f_i - \kappa_j \sum \kappa_i^2 f_i. \tag{4.2}
$$

Theorem 4.1. *Let Σ be a smooth strictly locally convex graph in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$ and*

$$
\nu^{n+1} \geq 2a > 0 \text{ on } \Sigma. \tag{4.3}
$$

For $\mathbf{x} \in \Sigma$ let $\kappa_{\max}(\mathbf{x})$ be the largest principal curvature of Σ at $\mathbf{x}$. Then

$$
\max_{\Sigma} \frac{\kappa_{\max}}{\nu^{n+1} - a} \leq \max \left\{ \frac{4n}{a^3}, \max_{\partial \Sigma} \frac{\kappa_{\max}}{\nu^{n+1} - a} \right\}. \tag{4.4}
$$

Proof. Let

$$
M_0 = \max_{\mathbf{x} \in \Sigma} \frac{\kappa_{\max}(x)}{\nu^{n+1} - a}. \tag{4.5}
$$

Assume $M_0 > 0$ is attained at an interior point $\mathbf{x}_0 \in \Sigma$. Let $\tau_1, \ldots, \tau_n$ be a local orthonormal frame around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ at $\mathbf{x}_0$. We may assume $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$. Thus, at $\mathbf{x}_0$, $\frac{h_{11}}{\nu^{n+1} - a}$ has a local maximum. Therefore,

$$
\frac{h_{11i}}{h_{11}} - \frac{\nabla_i \nu^{n+1}}{\nu^{n+1} - a} = 0, \tag{4.6}
$$

$$
\frac{h_{11ii}}{h_{11}} - \frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1} - a} \leq 0. \tag{4.7}
$$

Using (4.2), we find after differentiating the equation $F(h_{ij}) = \sigma$ twice that

Lemma 4.2. *At* $\mathbf{x}_0$*,*

$$F^{ii} h_{11ii} = -F^{ij,rs} h_{ij1} h_{rs1} + \sigma(1 + \kappa_1^2) - \kappa_1 \sum f_i - \kappa_1 \sum \kappa_i^2 f_i. \tag{4.8}$$

By Lemma 2.1 we immediately derive

Lemma 4.3. *Let Σ be a smooth hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$. Then in a local orthonormal frame,*

$$F^{ij} \nabla_{ij} \nu^{n+1} = \frac{2}{u} F^{ij} \nabla_i u \nabla_j \nu^{n+1} + \sigma(1 + (\nu^{n+1})^2) - \nu^{n+1} \Big(\sum f_i + \sum f_i \kappa_i^2 \Big). \tag{4.9}$$

Using Lemma 4.2 and Lemma 4.3 we find from (4.7)

$$\begin{aligned} 0 \geq & - F^{ij,rs} h_{ij1} h_{rs1} + \sigma \Big(1 + \kappa_1^2 - \frac{1 + (\nu^{n+1})^2}{\nu^{n+1} - a} \kappa_1 \Big) \\ & + \frac{a \kappa_1}{\nu^{n+1} - a} \Big(\sum f_i + \sum \kappa_i^2 f_i \Big) - \frac{2\kappa_1}{\nu^{n+1} - a} F^{ij} \frac{u_i}{u} \nabla_j \nu^{n+1}. \end{aligned} \tag{4.10}$$

Next we use an inequality due to Andrews [1] and Gerhardt [3] which states

$$- F^{ij,kl} h_{ij1} h_{kl,1} \geq \sum_{i \neq j} \frac{f_i - f_j}{\kappa_j - \kappa_i} h_{ij1}^2 \geq 2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} h_{i11}^2. \tag{4.11}$$

Recall that (see (2.18))

$$\nabla_i \nu^{n+1} = \frac{u_i}{u} (\nu^{n+1} - \kappa_i).$$

Then at $\mathbf{x}_0$, we obtain from (4.6)

$$h_{11i} = \frac{\kappa_1}{\nu^{n+1} - a} \frac{u_i}{u} (\nu^{n+1} - \kappa_i). \tag{4.12}$$

Inserting this into (4.11) we derive

$$- F^{ij,kl} h_{ij1} h_{kl,1} \geq 2 \Big(\frac{\kappa_1}{\nu^{n+1} - a} \Big)^2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1})^2. \tag{4.13}$$

Note that we may write

$$\sum f_i + \sum \kappa_i^2 f_i = (1 - (\nu^{n+1})^2) \sum f_i + \sum (\kappa_i - \nu^{n+1})^2 f_i + 2\sigma \nu^{n+1}. \tag{4.14}$$

Combining (4.11), (4.13) and (4.14) gives

$$
\begin{aligned}
0 \geq \sigma\Big(1 + \kappa_1^2 - &\frac{(1+(\nu^{n+1})^2)}{\nu^{n+1}-a}\kappa_1\Big) \\
&+ \frac{a\kappa_1}{\nu^{n+1}-a}\Big((1-(\nu^{n+1})^2)\sum f_i + \sum(\kappa_i - \nu^{n+1})^2 f_i + 2\sigma\nu^{n+1}\Big) \\
&+ 2\frac{\kappa_1}{\nu^{n+1}-a}\sum f_i \frac{u_i^2}{u^2}(\kappa_i - \nu^{n+1}) \\
&+ 2\frac{\kappa_1^2}{(\nu^{n+1}-a)^2}\sum_{i\geq 2}\frac{f_i - f_1}{\kappa_1 - \kappa_i}\frac{u_i^2}{u^2}(\kappa_i - \nu^{n+1})^2.
\end{aligned}
\tag{4.15}
$$

Note that (assuming $\kappa_1 \geq \frac{2}{a}$) all the terms of (4.15) are positive except possibly the ones in the sum involving $(\kappa_i - \nu^{n+1})$ and only if $\kappa_i < \nu^{n+1}$.

Therefore define

$$
\begin{aligned}
J &= \{i : \kappa_i - \nu^{n+1} < 0,\ f_i < \theta^{-1} f_1\}, \\
L &= \{i : \kappa_i - \nu^{n+1} < 0,\ f_i \geq \theta^{-1} f_1\},
\end{aligned}
$$

where $\theta \in (0,1)$ is to be chosen later. Since $\sum u_i^2/u^2 = |\tilde\nabla u|^2 = 1 - (\nu^{n+1})^2 \leq 1$ and $\kappa_1 f_1 \leq \sigma$, we have

$$
\sum_{i\in J}(\kappa_i - \nu^{n+1}) f_i \frac{u_i^2}{u^2} \geq -\frac{n}{\theta} f_1 \geq -\frac{n\sigma}{\theta\kappa_1}. \tag{4.16}
$$

Finally,

$$
\begin{aligned}
\frac{2\kappa_1^2}{\nu^{n+1}-a} &\sum_{i\in L}\frac{f_i - f_1}{\kappa_1 - \kappa_i}\frac{u_i^2}{u^2}(\kappa_i - \nu^{n+1})^2 \\
&\geq \frac{2(1-\theta)\kappa_1}{\nu^{n+1}-a}\sum_{i\in L}(\kappa_i - \nu^{n+1})^2 f_i \frac{u_i^2}{u^2} \\
&= -2\kappa_1 \sum_{i\in L} f_i \frac{u_i^2}{u^2}(\kappa_i - \nu^{n+1}) - \frac{2\theta\kappa_1}{\nu^{n+1}-a}\sum_{i\in L}(\kappa_i - \nu^{n+1})^2 f_i \frac{u_i^2}{u^2} \\
&\quad + \frac{2\kappa_1}{\nu^{n+1}-a}\sum_{i\in L} f_i \frac{u_i^2}{u^2}(\kappa_i^2 - (a+\nu^{n+1})\kappa_i + a\nu^{n+1}) \\
&\geq -2\kappa_1 \sum_{i\in L} f_i \frac{u_i^2}{u^2}(\kappa_i - \nu^{n+1}) - \frac{2\theta}{a}\kappa_1 \sum_{i\in L}(\kappa_i - \nu^{n+1})^2 f_i - \frac{6\sigma}{a}\kappa_1.
\end{aligned}
\tag{4.17}
$$

In deriving the last inequality in (4.17) we have used that $\kappa_i f_i \leq \sigma$ for each i and that $\nu^{n+1} \geq 2a$. We now fix $\theta = \frac{a^2}{4}$. From (4.16) and (4.17) we see that the right

hand side of (4.15) is strictly positive provided that $\kappa_1 > \frac{4n}{a^2}$, completing the proof of Theorem 4.1. □

5. Uniqueness and foliations

In this section we identify a class of curvature functions for which there is uniqueness. This implies that for these curvature functions and smooth asymptotic boundaries Γ which are Jordan, there is a foliation of each component of $\mathbb{H}^{n+1} \setminus \mathcal{C}(\Gamma)$ (the complement of the hyperbolic convex hull of Γ) by solutions $f(\kappa) = \sigma$ as σ varies between 0 and 1.

Theorem 5.1. *Let $f(\kappa)$ satisfy* (1.5)-(1.10) *in the positive cone K_n^+ and in addition satisfy*

$$\sum_i f_i > \sum \lambda_i^2 f_i \text{ in } K_n^+ \cap \{0 < f < 1\}. \tag{5.1}$$

Let Σ_i, $i = 1, 2$ be strictly locally convex hypersurfaces (oriented up) in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma_i \in (0, 1)$, $\sigma_1 \leq \sigma_2$, with the same boundary in the horosphere $x_{n+1} = \epsilon$ or with the same asymptotic boundary $\Gamma = \partial\Omega$. Then Σ_2 lies below Σ_1, that is, if Σ_i are represented as graphs $x_{n+1} = u_i(x)$ over $\Omega \subset R^n$, then $u_2 \leq u_1$ in Ω.

Proof. We build on an idea of Schlenker [12]. Suppose for contradiction that Σ_2 contains points in the unbounded region of $\mathbb{R}^{n+1}_+ \setminus \Sigma_1$ and let P be a point of Σ_2 farthest from Σ_1 (necessarily P is not a boundary point) where the maximal distance, say t^* is achieved. Then the local parallel hypersurfaces Σ_2^t to Σ_2 obtained by moving a distance t (on the concave side of Σ_2 near P) *are convex* and contact Σ_1 at a point Q in Σ_1 when $t = t^*$. Moreover $\Sigma_2^{t^*}$ *locally lies below* Σ_1 by the maximality of the distance t^*. We claim that the distance function $d(x, \Sigma_2)$ is smooth in a neighborhood of Q. To show this we need only show (see [9]) that P is the unique closest point to Q on Σ_2. If P' was a second point of Σ_2 at distance t^* from Q, then the local parallel hypersurfaces Σ_2^t to Σ_2 obtained by moving a distance t (on the concave side of Σ_2 near P') are also convex and when $t = t^*$, contact Σ_1 at Q *and also locally lies below* Σ_1 by the previous argument. This is clearly impossible since Σ_1 has a unique tangent plane at Q.

The principal curvatures of Σ_2^t at points along the normal geodesic emanating from any point of Σ_1 (say near P) are given by the ode (see [4]):

$$\kappa_i'(t) = 1 - \kappa_i^2.$$

In particular, if $\kappa_i(0) < 1$, then $\kappa_i(0) \le \kappa_i(t) < 1$ while if $\kappa_i(0) > 1$, then $1 < \kappa_i(t) \le \kappa_i(0)$. Of course if $\kappa_i(0) = 1$, then $\kappa_i(t) \equiv 1$. Moreover by (5.1),

$$\frac{d}{dt} f(\kappa)(t) = \sum f_i - \sum k_i^2 f_i > 0 \text{ in } K_n^+ \cap \{0 < f < 1\}. \tag{5.2}$$

It follows that the Σ_2^t satisfy $f(\kappa) > \sigma_2$ and so are strict subsolutions of the equation $f(\kappa) = \sigma_1$. On the other hand at $t = t*$ we have Σ_2^t lies below Σ_1 but touches Σ_1 at Q violating the maximum principle. □

Corollary 5.2. *Let $f(\kappa)$ satisfy* (1.5)-(1.10) *in the positive cone K_n^+ and in addition satisfy* (5.1). *Let Σ_i, $i = 1, 2$ be strictly locally convex graphs (oriented up) in $\mathbb{H}^{n+1}$over $\Omega \subset \mathbb{R}^n$ satisfying $f(\kappa) = \sigma \in (0, 1)$ with the same boundary in the horosphere $x_{n+1} = \epsilon$ or with the same asymptotic boundary $\Gamma = \partial\Omega$. Then $\Sigma_1 = \Sigma_2$.*

Example 5.3. For $l - 1$ or $l - 2$, let $f = (\frac{H_n}{H_l})^{\frac{1}{n-l}}$ in the cone $K_n^+ \subset \mathbb{R}^n$. Then (see Lemma 2.14 of [16])

$$f_i = \frac{f}{n-l}\Big(\frac{1}{\lambda_i} - (\log H_l)_i\Big)\frac{f}{n-l}\Big(\frac{1}{\lambda_i} - \frac{H_{l-1;i}}{H_l}\Big),$$

where $H_{l-1;i} = H_{l-1}|_{\lambda_i=0}$. Hence,

$$\sum f_i = \frac{f}{n-l}\Big(n\frac{H_{n-1}}{H_n} - l\frac{H_{l-1}}{H_l}\Big) \ . \tag{5.3}$$

Similarly,

$$\sum \lambda_i^2 f_i = \frac{f}{n-l}\Big(nH_1 - \frac{\sum \lambda_i^2 H_{l-1;i}}{H_l}\Big) \ .$$

Using

$$\frac{\sum \lambda_i^2 H_{l-1;i}}{H_l} = nH_1 - (n-l)\frac{H_{l+1}}{H_l} \ ,$$

we find

$$\sum \lambda_i^2 f_i = f\frac{H_{l+1}}{H_l} \ . \tag{5.4}$$

Combining (5.3) and (5.4) gives

$$\sum f_i - \sum \lambda_i^2 f_i = \frac{f}{n-l}\Big(n\frac{H_{n-1}}{H_n} - l\frac{H_{l-1}}{H_l} - (n-l)\frac{H_{l+1}}{H_l}\Big). \tag{5.5}$$

By the Newton-Maclaurin inequalities,

$$\frac{H_{n-1}}{H_n} \geq \frac{H_{l-1}}{H_l}$$

with equality if and only all the λ_i are equal. Hence,

$$\sum f_i - \sum \lambda_i^2 f_i \geq f\Big(\frac{H_{n-1}}{H_n} - \frac{H_{l+1}}{H_l}\Big)\ . \tag{5.6}$$

Therefore if $l-1$, we find

$$\sum f_i - \sum \lambda_i^2 f_i \geq 1 - f^2 > 0 \text{ in } K_n^+ \cap \{0 < f < 1\} \tag{5.7}$$

while if $l-2$ we similarly find

$$\sum f_i - \sum \lambda_i^2 f_i \geq \frac{H_{n-1}}{H_n} f(1-f^2) \geq 1 - f^2 > 0 \text{ in } K_n^+ \cap \{0 < f < 1\}. \tag{5.8}$$

We now complete the **proof of Theorem 1.7**.

Proof. a. For Γ smooth and $f(\kappa)$ satisfying the conditions of Theorem 5.1, we have by Theorem 1.2 and Theorem 5.1 a smooth "monotone decreasing" family of smooth solutions $\Sigma^\sigma = \text{graph } \ u^\sigma(x)$, $x \in \Omega$ of (1.1), (1.2). That is, if $\sigma_1 < \sigma_2$, then $u^{\sigma_1} > u^{\sigma_2}$ in Ω. Note also that if $\Omega \subset B_\delta(0)$ then

$$u^\sigma < v^\sigma(x) := -\frac{\sigma\delta}{\sqrt{1-\sigma^2}} + \sqrt{\frac{\delta^2}{1-\sigma^2} - |x|^2} \text{ in } \ \Omega,$$

where $v^\sigma(x)$ corresponds to the equidistant sphere solution of $f(\kappa) = \sigma$, which is a graph over $B_\delta(0)$. As $\sigma \to 1$, $v(x) \to 0$ uniformly and so the same holds for $u^\sigma(x)$.

We claim that as $\sigma \to 0$, Σ^σ tends to the component S of $\partial\mathcal{CH}(\Gamma)$ that is a graph over Ω. To see this note that Σ^σ lies below S but also eventually lies above any smooth strictly locally convex hypersurface S' by Theorem 5.1.

This completes the proof of Theorem 1.7 part a. In order to prove part b, it suffices by a standard approximation argument, to show that the graph solutions of $f(\kappa) = \sigma$ have uniformly bounded principal curvatures on compact subdomains of Ω, independent of the smoothness of Γ. We carry this out for the special curvature quotients $f = \frac{H_n}{H_{n-1}}$ in Lemma 5.4 below, thus completing the proof of part b. $\square$

Lemma 5.4. *Let $\Sigma = \{graph\ u(x) : x \in \Omega\}$ be the unique strictly locally convex solution of $\frac{H_n}{H_{n-1}}(\kappa) = \sigma \in (0,1)$. For any compact subdomain $\Omega' \subset\subset \Omega$, let $\Sigma' = \{graph\ u(x) : x \in \Omega'\}$. Then,*

$$\max_{\mathbf{x}\in\Sigma'} \kappa_{\max} \le C,$$

where C depends only on σ and the (Euclidean) distance from Ω' to $\partial\Omega$.

Proof. Fix a small constant $\theta \in (0,1)$ and set $\phi = (u-\theta)_+$. Recall from Lemma 2.1,

$$Lu = \frac{2}{u}F^{ij}u_iu_j + \sigma u\nu^{n+1} - u\sum f_i \tag{5.9}$$

We modify the argument of section 4 by setting

$$M_0 = \max_{\mathbf{x}\in\Sigma} \phi\,\kappa_{\max}(x)\ . \tag{5.10}$$

Then $M_0 > 0$ is attained at an interior point $\mathbf{x}_0 \in \Sigma$. Let $\tau_1, \ldots, \tau_n$ be a local orthonormal frame around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i\delta_{ij}$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ at $\mathbf{x}_0$. We may assume $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$. Thus, at $\mathbf{x}_0$, $\log\phi + \log h_{11}$ has a local maximum and so,

$$\frac{\phi_i}{\phi} + \frac{h_{11i}}{h_{11}} = 0, \tag{5.11}$$

$$\frac{\phi_{ii}}{\phi} + \frac{h_{11ii}}{h_{11}} \le 0. \tag{5.12}$$

As in section 4 we obtain from (5.12) and (5.9) that at $\mathbf{x}_0$,

$$0 \ge -\frac{u\kappa_1}{\phi}\sum f_i + \sigma(1+\kappa_1^2) - \kappa_1\sum f_i - \kappa_1\sum \kappa_i^2 f_i. \tag{5.13}$$

From the calculations of Example 5.3,

$$1 \le \sum f_i \le n,\ \ \sum \kappa_i^2 f_i = \sigma^2\ . \tag{5.14}$$

Hence from (5.13) and (5.14) we obtain $\phi\kappa_1 \le C$. Choosing θ so small that $u \ge 2\theta$ on Ω' completes the proof. $\square$

References

[1] B. Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. PDE **2** (1994), 151–171.

[2] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations III: Functions of eigenvalues of the Hessians*, Acta Math. **155** (1985), 261–301.

[3] C. Gerhardt, *Closed Weingarten hypersurfaces in Riemannian manifolds*, J. Differential Geom. **43** (1996), 612–641.
[4] A. Gray, *Tubes*, Addison Wesley Publising Company, 1990.
[5] B. Guan and J. Spruck, *Hypersurfaces of constant mean curvature in hyperbolic space with prescribed asymptotic boundary at infinity*, Amer. J. Math. **122** (2000), 1039–1060.
[6] B. Guan and J. Spruck, *Hypersurfaces of constant curvature in Hyperbolic space II*, J. European Math. Soc. **12** (2010), 797–817.
[7] B. Guan, J. Spruck and M. Szapiel, *Hypersurfaces of constant curvature in Hyperbolic space I.* J. Geom. Anal. **19** (2009), 772–795.
[8] F. Labourie, *Problème de Minkowski et surfaces à courbure constante dan les variétés hyperboliques*, Bull. Soc. Math. Fr. **119** (1991), 307–325.
[9] Y.-Y. Li and L. Nirenberg, *Regularity of the distance function to the boundary*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **29** (2005), 257–264.
[10] B. Nelli and J. Spruck, *On existence and uniqueness of constant mean curvature hypersurfaces in hyperbolic space*, Geometric Analysis and the Calculus of Variations, Internat. Press, Cambridge, MA, 1996, 253–266.
[11] H. Rosenberg, and J. Spruck, *On the existence of convex hypersurfaces of constant Gauss curvature in hyperbolic space*, J. Differential Geom. **40** (1994), 379–409.
[12] J-M. Schlenker, *Réalisations de surfaces hyperboliques complètes dans H^3*, Annales de l'institut Fourier, **48** (1998), 837–860.
[13] G. Smith, *Moduli of flat conformal structures of hyperbolic type*, ArXiv 0804.0744v6 (2009).
[14] G. Smith, *Lagrangian curvature*, ArXiv 0506230v3 (2005).
[15] G. Smith, *The Plateau problem for general curvature functions*, arXiv:1008.3545 (2010)
[16] J. Spruck, *Geometric aspects of the theory of fully nonlinear elliptic equations*, Global Theory of Minimal Surfaces, Clay Mathematics Proceedings Vol. 3 (2005), 283–309.

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